The primitivity of operators in the algebra of binary relations under conjunctions of containments

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KNOWLEDGE IN ACTION

Algebra of binary relations

Operations on binary relations over some domain V [Peirce, Schröder, Tarski]

$$
\begin{aligned}
id &= \{(x,x) \mid x \in V\} \\
r^{-1} &= \{(x,y) \mid (y,x) \in r\} \\
r \circ s &= \{(x,y) \mid \exists z : (x,z) \in r \wedge (z,y) \in s\} \\
r \cup s &= \{(x,y) \mid (x,y) \in r \vee (x,y) \in s\} \\
r - s &= \{(x,y) \mid (x,y) \in r \wedge (x,y) \notin s\} \\
r \cap s &= \{(x,y) \mid (x,y) \in r \wedge (x,y) \in s\}
\end{aligned}
$$

Additional Operations

$$\pi_1(r) = \{(x,x) \mid \exists y : (x,y) \in r\} \qquad \text{(provide tests)}$$
$$\pi_2(r) = \{(x,x) \mid \exists y : (y,x) \in r\}$$
$$\overline{\pi}_1(r) = \{(x,x) \mid \neg(\exists y) : (x,y) \in r\} \qquad \text{(provide negative tests)}$$
$$\overline{\pi}_2(r) = \{(x,x) \mid \neg(\exists y) : (y,x) \in r\}$$
$$\mathit{all} = V^2$$
$$\mathit{di} = \mathit{all} - \mathit{id}$$
$$r^+ = \text{the transitive closure of } r$$

Fragments & Path Queries

The most basic fragment $\mathcal{N}$ is the semiring $\{\emptyset, id, \cup, \circ\}$

$\mathcal{N}(F)$ denotes $\mathcal{N}$ extended with the operations in F

Fix a binary relational vocabulary Γ (Label set)

Expressions over a fragment F are built from relation names in Γ using the operations in F

$\Rightarrow$ Map instances of Γ (graphs) to binary relations (*path query*)

Write $F_1 \leq F_2$ if every path query expressible in $\mathcal{N}(F_1)$ is also expressible in $\mathcal{N}(F_2)$

Well established logic: $\mathcal{N}(^{-1}, -, di) \equiv \mathrm{FO}^3$ (Tarski & Givant)

Examples

Consider Hobbies(person,hname), Person(pname,age)

"Output the persons that share a hobby"

- $(\text{Hobbies} \circ (\text{Hobbies}^{-1})) - \mathit{id}$

"Output the persons that do not have any hobbies"

- $\overline{\pi}_1(\text{Hobbies}) \circ \pi_1(\text{Person})$

Interdependencies Between Features

Some operators can be constructed with other operators:

$$
\begin{aligned}
all &\equiv di \cup id \\
di &\equiv all - id \\
e_1 \cap e_2 &\equiv e_1 - (e_1 - e_2) \\
\pi_1(e) &\equiv (e \circ e^{-1}) \cap id \equiv (e \circ all) \cap id \equiv \overline{\pi}_1(\overline{\pi}_1(e)) \\
\pi_2(e) &\equiv (e^{-1} \circ e) \cap id \equiv (all \circ e) \cap id \equiv \overline{\pi}_2(\overline{\pi}_2(e)) \\
\overline{\pi}_1(e) &\equiv id - \pi_1(e) \\
\overline{\pi}_2(e) &\equiv id - \pi_2(e)
\end{aligned}
$$

⚠ Operator can be expressed in $\mathcal{N}(F)$ without belonging to F

Completion of F

Define $\overline{F}$ as the smallest superset of F that satisfy the *completion axioms*:

- $di \in \overline{F}$, then $all \in \overline{F}$
- $\overline{\pi} \in \overline{F}$, then $\pi \in \overline{F}$
- $\cap \in \overline{F}$ and $di \in \overline{F}$, then $\pi \in \overline{F}$
- $\cap \in \overline{F}$ and $^{-1} \in \overline{F}$, then $\pi \in \overline{F}$
- $- \in \overline{F}$ and $\pi \in \overline{F}$, then $\overline{\pi} \in \overline{F}$
- $- \in \overline{F}$, then $\cap \in \overline{F}$

Example: $\overline{\{all, -\}} = \{di, all, \cap, -, \pi, \overline{\pi}\}$

Clearly, $\mathcal{N}(F) \equiv \mathcal{N}(\overline{F})$

Primitivity for path queries

An operator g is *primitive* (for path queries) if for every fragment F with $g \notin \overline{F}$: $\{g\} \not\leq F$

Theorem ([Fletcher et al., 2011])

Every operator is primitive for path queries

$\Rightarrow$ The completion axioms are complete for the purpose of comparing fragments

Nonemptiness

A Boolean query maps graphs to yes/no

Typically expressed by nonemptiness expressions $e \neq \emptyset$
Example: $R^2 - R \neq \emptyset$ (nontransitivity)

Define $F^{\neq\emptyset}$ as the set of boolean queries expressible in the form $e \neq \emptyset$ with $e \in \mathcal{N}(F)$

Primitivity for nonemptiness

Converse and Transitive closure are not primitive for nonemptiness:

Theorem ([Fletcher et al., 2011, 2013])

If $\overline{F}$ has projection, but neither intersection nor transitive closure, then

$$F^{\neq\emptyset} \subseteq F - \{^{-1}\}^{\neq\emptyset}$$

If $F \subseteq \{\pi, di, ^{+}\}$ and Γ only contains one edge label, then

$$F^{\neq\emptyset} \subseteq F - \{^{+}\}^{\neq\emptyset}$$

Examples:

- $R^2 \circ R^{-1} \circ R^3 \neq \emptyset \;\equiv\; \pi_1(R^2 \circ \pi_2(\pi_1(R^3) \circ R)) \neq \emptyset$
- $\pi_1(R^3) \circ R^{+} \circ di \circ \pi_2(R) \circ R^2 \neq \emptyset \;\equiv\; \pi_1(R^3) \circ (R \cup R^2) \circ di \circ \pi_2(R) \circ R^2 \neq \emptyset$

Conjunctions of Containments

Nonemptiness is not the only way for expressing boolean queries

- Example: consider $e = \emptyset$ statements

Natural to express properties as finite conjunctions of containment statements

- A containment statement $e_1 \subseteq e_2$ holds on G if $e_1(G) \subseteq e_2(G)$

Examples:

- Total order: $R^2 \subseteq R \ \wedge \ id \subseteq R \ \wedge \ R \cap R^{-1} \subseteq id \ \wedge \ all \subseteq R \cup R^{-1}$
- Function: $R^{-1} \circ R \subseteq id$
- Connectivity: $all \subseteq (R \cup R^{-1})^+$

Main result

For a fragment F define $F^{\subseteq}$ as the boolean queries expressible by finite conjunctions of containment statements in $\mathcal{N}(F)$

Theorem

Every operator is primitive for finite conjunctions of containments

In other words, if $g \notin \overline{F}$ then $\{g\}^{\subseteq} \not\subseteq F^{\subseteq}$

$\Rightarrow$ The completion axioms are complete for comparing fragments under finite conjunctions of containments

Fragments with Difference

Notice that $e_1 \subseteq e_2 \ \wedge \ e_3 \subseteq e_4 \ \equiv \ e_1 - e_2 \cup e_3 - e_4 = \emptyset$

$\Rightarrow$ emptiness $\equiv$ containment when difference is present

When difference is present inexpressibility sometimes reduces to inexpressibility for (non)emptiness

- Example: $\{\pi\}^{\subseteq} \not\subseteq \{-, ^{+}\}^{\subseteq}$ reduces to $\{\pi\}^{=\emptyset} \not\subseteq \{-, ^{+}\}^{=\emptyset}$

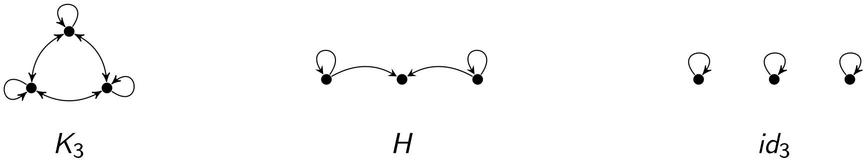
Primitivity of Projection

The only fragment of interest is $\{di, ^{-1}, ^{+}\}$

"Paths of length 2 only start in loops" is not expressible in $\{di, ^{-1}, ^{+}\}^{\subseteq}$

- Expressed by $R \circ \pi_1(R) \subseteq id$ in $\{\pi\}^{\subseteq}$

We prove that queries in $\{di, ^{-1}, ^{+}\}^{\subseteq}$ cannot be false on K_3, true on H and true on id_3 simultaneously



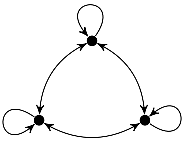
Primitivity of Coprojection

The only fragment of interest is $\{di, \cap, ^{-1}, ^{+}\}$

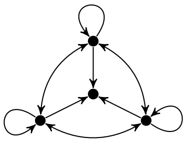
"Every node is at most one step away from a sink node" is not expressible in $\{di, \cap, ^{-1}, ^{+}\}^{\subseteq}$

- Expressed by $\pi_1(R) \subseteq \pi_1(R \circ \overline{\pi}_1(R))$ in $\{\overline{\pi}\}^{\subseteq}$

We prove that queries in $\{di, \cap, ^{-1}, ^{+}\}^{\subseteq}$ cannot be false on K_3 and true on H simultaneously



K_3



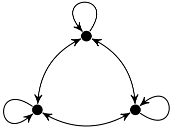
H

Primitivity of Difference

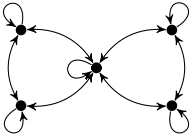
The only fragment of interest is $\{di, \overline{\pi}, ^{-1}, \cap, ^{+}\}$

The query $id \subseteq R^2 \circ (R^2 - R) \circ R^2$ is not expressible in $\{di, \overline{\pi}, ^{-1}, \cap, ^{+}\}^{\subseteq}$

We prove that queries in $\{di, \overline{\pi}, ^{-1}, \cap, ^{+}\}^{\subseteq}$ cannot be false on K_3 and true on B simultaneously



K_3



B

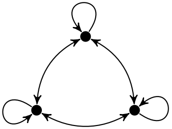
Primitivity of Intersection

The only fragment of interest is $\{di, \overline{\pi}, {}^{-1}, {}^{+}\}$

"The only transitive edges are loops" is not expressible in $\{di, \overline{\pi}, {}^{-1}, {}^{+}\}^{\subseteq}$

- Expressed by $R^2 \cap R \subseteq id$ in $\{\cap\}^{\subseteq}$

We prove that queries $\{di, \overline{\pi}, {}^{-1}, {}^{+}\}^{\subseteq}$ cannot be false on K_3, true on id_3 and true on ℓ_2 simultaneously



K_3

id_3

ℓ_2

Primitivity of Transitive Closure

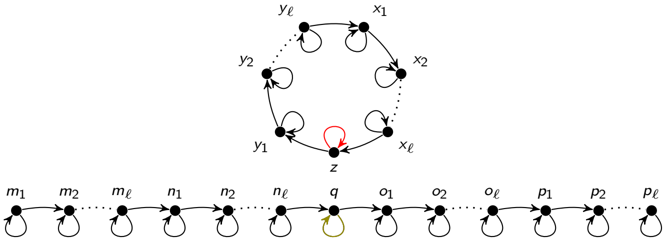
The fragment of interest is $\{^{-1}, di, -\}$

"There is a cycle through every node" is not expressible in $\{^{-1}, di, -\}^{\subseteq}$

- Expressed by $id \subseteq R^{+}$ in $\{^{+}\}^{\subseteq}$

⚠ In contrast with the collapses of $^{+}$ under nonemptiness!

We show that the families of graphs G_1^{ℓ} and G_2^{ℓ} are not distinguishable in FO



Primitivity of *all*

The fragment of interest is $\{^{-1}, -, ^{+}\}$

"The graph is complete" is not expressible in $\{^{-1}, -, ^{+}\}^{\subseteq}$

- Expressed by $all \subseteq R$ in $\{all\}^{\subseteq}$

We prove that queries in $\{^{-1}, -, ^{+}\}^{\subseteq}$ cannot be true on a single self-loop and false on two disjoint self-loops simultaneously

id_1

id_2

Primitivity of Diversity

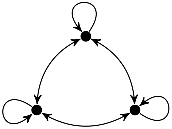
The only fragment of interest is $\{^{-1}, \cap, all, \overline{\pi}, ^{+}\}$

"There is only one node" is not expressible in $\{^{-1}, \cap, all, \overline{\pi}, ^{+}\}^{\subseteq}$

- Expressed by $di \subseteq \emptyset$ in $\{di\}^{\subseteq}$

We prove that queries in $\{^{-1}, \cap, all, \overline{\pi}, ^{+}\}^{\subseteq}$ cannot be true on id_1 and false on K_3 simultaneously

id_1



K_3

Primitivity of Converse

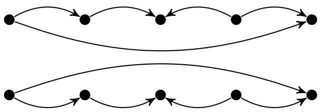
The fragment of interest is $\{di, -, ^+\}$

"If there is a path $a \to b \to c \leftarrow d \to e$ with $d \neq b$ then $a \to e$" is not expressible in $\{di, -, ^+\}$

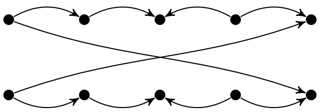
- Expressed by $R^2 \circ R^{-1} \circ R \subseteq R^2 \cup R$ in $\{^{-1}\}^{\subseteq}$

⚠ In contrast with the collapses of $^{-1}$ under nonemptiness!

We prove that queries in $\{di, -, ^+\}^{\subseteq}$ cannot be true on G_1 and false on G_2 simultaneously



G_1



G_2

Directions for further research

- All of our separating queries do not use conjunction
  - ? When does conjunction actually add expressive power
- Compare the expressive power of nonemptiness and containment
  - ⇒ Largely completed this work, one open question remains:
    $\{\pi\}^{\neq\emptyset} \overset{?}{\subseteq} \{di, ^{-1}, ^{+}\}^{\subseteq}$
- Another interesting derived operator: Residuations [Pratt. Origins of calculus of binary relations]
  C/B is the maximal X such that

$$X \circ B \subseteq C$$

  $B\backslash C$ is the maximal X such that

$$B \circ X \subseteq C$$